Integral conditions for curves to be simple or unknotted

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Abstract

By considering integral bounds on the Schwarzian derivative we extend previous results on sufficient conditions for curves in euclidean spaces to be simple or unknotted. The conditions are optimal.

1. Introduction

Ahlfors' Schwarzian derivative for parametrized curves in $\mathbb{R}^n$ [Ah] is a useful concept in the formulation of sufficient conditions for curves to be simple or unknotted [ChG], [Ch]. Recent results on the univalence of conformal parametrizations of minimal surfaces in $\mathbb{R}^3$ and holomorphic immersions of planar domains into $\mathbb{C}^n$ appeal to this operator in a crucial manner [ChDO1], [ChDO2]. Injectivity in this context is ultimately a consequence of the disconjugacy of a linear equation of order two that involves a pointwise upper bound of the Schwarzian. The purpose of this paper is to establish new criteria of this type by considering instead integral bounds of the Schwarzian. In Theorems 1 and 2 below we deal with conditions that prevent a curve from self-intersecting or from becoming a knot. They are sharp in the sense that there exist parametrizations of non-simple or knotted curves that fail to satisfy the respective criterion by arbitrarily small amounts. The condition for simple curves involving a pointwise bound is satisfied by *extremal* parametrizations $f : (a, b) \to \mathbb{R}^n$ that reflect sharpness in the additional sense that $f([a, b])$ is closed. Corresponding extremal parametrizations for Theorem 1 fail to be smooth and exhibit a Schwarzian derivative equal to a delta function.

2. Ahlfors' Schwarzian

Let $f : (a, b) \to \mathbb{R}^n$ be a C^3 curve with $f' \neq 0$, and let $X \cdot Y$ stand for the euclidean inner product vectors X, Y in $\mathbb{R}^n$ and $|X|^2 = X \cdot X$. Ahlfors considered separately generalizations $S_1 f$ and $S_2 f$ of the real and imaginary parts of the usual Schwarzian in complex analysis [Ah]. So far, only $S_1 f$ has played a role for questions of injectivity. In [ChG] we showed that the real part of Ahlfors' Schwarzian, defined by

$$S_1 f = \frac{f' \cdot f'''}{|f'|^2} - 3 \frac{(f' \cdot f'')^2}{|f'|^4} + \frac{3}{2} \frac{|f''|^2}{|f'|^2} \, ,$$

can be written in terms of the velocity $v = |f'|$ and the curvature k of the trace of f as

$$S_1 f = \left(\frac{v'}{v}\right)' - \frac{1}{2}\left(\frac{v'}{v}\right)^2 + \frac{1}{2}v^2 k^2 \,. \tag{1}$$

This expression is invariant under the Möbius transformations of $\mathbb{R}^n \cup \{\infty\}$. Observe that $S_1 f = Ss + \frac{1}{2}v^2 k^2$, where $s = s(x)$ is arc length as a function of $x \in (a, b)$. Here, for a real valued function h with $h' \neq 0$, Sh is the usual Schwarzian

$$Sh = \left(\frac{h''}{h'}\right)' - \frac{1}{2}\left(\frac{h''}{h'}\right)^2 \,.$$

For a change of parameter $x = h(y)$,

$$S_1(f \circ h) = [(S_1 f) \circ h](h')^2 + Sh \,. \tag{2}$$

We recall the main results in [ChG], [Ch]:

Theorem A: *Let $p = p(x)$ be a continuous real-valued function on an open interval I such that any nontrivial solution of $u'' + pu = 0$ has at most one zero on I. Let $f : I \to \mathbb{R}^n$ be a C^3 curve with $f' \neq 0$. If $S_1 f \leq 2p$, then f is one-to-one on I and admits a spherically continuous extension to the closed interval, which is also one-to-one unless the trace of f is a circle, in which case $S_1 f \equiv 2p$.*

Theorem B: *Let $f : [-1, 1) \to \mathbb{R}^3$ parametrize a simple closed curve in $\mathbb{R}^3$. If the periodic continuation of f is C^3 and $S_1 f(x) \leq 2\pi^2$ for all $x \in (-1, 1)$, then $f([-1, 1))$ is unknotted.*

When formulating Theorem B on an interval $[a, b)$, the resulting upper bound after the linear change of parameter becomes $8\pi^2/(b-a)^2$. Let $l(I)$ be the length of an interval I and let $^+$ denote positive part.

Theorem 1: *Let I be an open interval, and let $f : I \to \mathbb{R}^n$ a C^3 curve with $f' \neq 0$. If*

$$\int_I (S_1 f)^+ (x)dx \ \leq \ \frac{8}{l(I)} \tag{3}$$

then f admits a spherically continuous extension to the closed interval $\bar{I}$ and $f(\bar{I})$ is simple.

Theorem 2: *Let $J = [a, b)$, and let $f : J \to \mathbb{R}^3$ parametrize a simple closed curve in $\mathbb{R}^3$. If the periodic continuation of f is C^3 and*

$$\int_J (S_1 f)^+ (x)dx \ \leq \ \frac{32}{l(J)} \tag{4}$$

then $f(J)$ is unknotted.

In Theorem 1, for any $\epsilon > 0$ there is a parametrization f with $\int (S_1 f)^+ dx < 8/l(I) + \epsilon$ for which $f(I)$ is not simple, and similarly for Theorem 2. Examples will be provided in the last section.

3. Proofs

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Let $p(x) \geq 0$ and $u(x)$ be a solution of $u'' + pu = 0$ with initial conditions, say at $x = 0$, $u(0) = 0, u'(0) = 1$. If $u(x) \geq 0$ for $0 \leq x \leq x_2$ then $u(x) \leq x$ because it is concave. By integrating twice we see that for such x

$$u(x) \; = \; x - \int_0^x (x-t)p(t)u(t)dt \; \geq \; x - \int_0^x t(x-t)p(t)dt \; \geq \; x - \frac{x^2}{4} \int_0^x p(t)dt. \quad (5)$$

The following result is well known (see, *e.g.*, [H, Corollary 5.1]). For convenience of the reader we include a brief proof.

Lemma 1: *Let $p = p(x)$ be continuous on (a, b), and suppose the equation*

$$u'' + pu = 0 \quad (6)$$

admits a non-trivial solution with two zeros in (a, b). Then

$$\int_a^b p^+(x)dx > \frac{4}{b-a}. \quad (7)$$

Proof: Suppose $u \not\equiv 0$ is a solution of (6) with consecutive zeros $a < x_1 < x_2 < b$. Because $p \leq p^+$, the Sturm comparison theorem guarantees the existence of a non-trivial solution of the equation $u'' + p^+ u = 0$ that vanishes twice. In other words, we may assume that the coefficient in (6) is non-negative. Also, it is sufficient to limit attention to the case in which $x_1 = 0$ and $u'(x_1) = 1$. For $x \in J = (0, x_2)$ we have from (5) that

$$u(x) \; \geq \; x - \frac{A}{4} x^2 \quad ,$$

where $A = \int_0^{x_2} p(x)dx$. We have used that $t(x-t) \leq x^2/4$. Therefore

$$\frac{A}{4} x_2^2 - x_2 \; \geq \; -u(x_2) = 0 \,,$$

which implies that

$$\int_a^b p(x)dx \; \geq \; A \; = \; \int_0^{x_2} p(x)dx \; \geq \; \frac{4}{x_2} \; > \; \frac{4}{b-a} \,.$$

The proof shows that the extremal situation occurs when the entire mass of p is concentrated exactly at the midpoint between both zeros. Indeed, by letting p be a delta function at the origin, one obtains zeros at ± 2 for the solution $u(x) = 2 - |x|$ of (6) (in the appropriate weak sense).

Proof of Theorem 1: Without loss of generality we may assume that $a = -b$ and that $|f'(0)| = 1$. Because the analysis of extension is symmetric at both endpoints, it is enough to show the extension on $(0, b]$. Let $p = f(0)$ and consider the curve $\phi = T \circ f$ for the Möbius transformation

$$T(q) = \frac{q - p}{|q - p|^2} \,.$$

Then $S_1\phi = S_1 f$, and it was shown in [ChG] that the function

$$u = |\phi'|^{-1/2} = \frac{|f - p|}{\sqrt{|f'|}}$$

satisfies

$$u'' + \frac{1}{2}(S_1 f)u \geq 0,$$

and therefore also

$$u'' + \frac{1}{2}(S_1 f)^+ u \geq 0,$$

whenever $f \neq p$. The initial conditions are $u(0) = 0$ and $u'(0) = 1$. On $[0, b)$ consider the solution w of

$$w'' + \frac{1}{2}(S_1 f)^+ w = 0 \quad, \quad w(0) = 0 \,, \, w'(0) = 1 \,. \tag{8}$$

It follows from (5) that for $x \in (0, b)$

$$w(x) \geq x - \frac{x^2}{2b} > 0,$$

with the same inequality holding for $u(x)$ by the Sturm comparison. We conclude that for $x \geq b/2$

$$u(x) \geq \frac{3b}{2},$$

and hence

$$\int_{b/2}^{b} u^{-2}(x)dx < \infty.$$

Therefore $\phi((b/2, b))$ has finite length, which shows that ϕ admits a continuous extension to $x = b$.

If $f(\bar{I})$ is not simple, then $f(\alpha) = f(\beta) = p$ for some $a \leq \alpha < \beta \leq b$. We may choose $\beta - \alpha$ smallest possible, so that f is injective on (α, β). Note that

$$\int_{\alpha}^{\beta} (S_1 f)^+ dx \leq \int_{a}^{b} (S_1 f)^+ dx \leq \frac{8}{b - a} \leq \frac{8}{\beta - \alpha},$$

which means that, after relabeling, we may assume that $a = \alpha$ and $b = \beta$. Consider the composition $\psi = T \circ f$ with the Möbius inversion

$$T(q) = \frac{q - p}{|q - p|^2}.$$

Then $S_1\psi = S_1 f$ and $\psi(a) = \psi(b) = \infty$. Again $u = |\psi'|^{-1/2}$ satisfies

$$u'' + \frac{1}{2}(S_1 f)^+ u \geq 0. \tag{9}$$

Fix $c \in (a, b)$ and let w be the solution on (a, b) of

$$w'' + \frac{1}{2}(S_1 f)^+ w = 0 \quad, \quad w(c) = u(c) \,, \, w'(c) = u'(c) \,. \tag{10}$$

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Lemma 1 shows that w cannot have two zeros on (a, b), and thus simple arguments based on the convexity or concavity of w show that it has well-defined limits at both endpoints a, b. If $w(d) = 0$ at some interior point, say $d < c$, then the argument presented in the previous paragraph shows that $w(x) > 0$ for $x \in (d, b)$ and $w(x) \geq \rho > 0$ near $x = b$. By Sturm comparison, $u(x) \geq w(x)$ on (c, b), and thus

$$\int_c^b u^{-2}(x) dx < \infty,$$

contradicting that $\psi(b) = \infty$.

Suppose, on the other hand, that $w(x) > 0$ on (a, b). If $w(a) > 0$ then $\psi(a)$ is finite, a contradiction. If $w(a) = 0$, we see from (5) that, near $x = b$, $w(x)$ will be bounded below by some positive constant because $(S_1 f)^+$ is a continuous function. Again we reach a contradiction. This proves the theorem.

Proof of Theorem 2: The proof follows that of Theorem B. The key step is to show that if $\Gamma = f([a, b])$ is a knot, then it can be laid out to form a planar, closed, non-simple curve for which the real part of the Schwarzian has not increased. This procedure is based on the results of Brickell and Hsiung [BH] in the course of their proof of the Fary-Milnor theorem.

For $p \in \mathbb{R}^3$, let C_p denote the *shell* of Γ with vertex p defined as the (developable) surface consisting of all segments $[p, q]$ with $q \neq p$ on Γ. The *indicatrix* I_p of C_p is the curve on the sphere $\mathbb{S}^2$ traced by the vectors $(q - p)/|q - p|$; its length $l(I_p)$ is called the total angle of I_p. A key result established in [BH] is that Γ is unknotted if $l(I_p) < 3\pi$ for all $p \in \Gamma$ (see also [Ch]). Therefore, if Γ is a knot then there is a point $p \in \Gamma$ for which $l(I_p) \geq 3\pi$. The cases $l(I_p) > 3\pi$ and $l(I_p) = 3\pi$ must be dealt with separately. Suppose first that $l(I_p) > 3\pi$. As the point p moves to a point p' slightly away from Γ, the number $l(I_{p'})$ varies continuously, except for jump increment in π. It follows that there exists $p' \notin \Gamma$ for which $l(I_{p'}) > 4\pi$. Since $l(I_r)$ is a continuous function of $r \in \mathbb{R}^3 \setminus \Gamma$ and since $l(I_r) \to 0$ as $|r| \to \infty$, there exists $p_0 \notin \Gamma$ for which $l(I_{p_0}) = 4\pi$. As was shown in [Ch], the shell C_{p_0} can be laid out isometrically onto the plane in a way that Γ traces out a C^3 closed curve γ that winding number 2 with respect to the point in the plane corresponding to the vertex p_0

In the case $l(I_p) = 3\pi$ we lay out Γ as before, with p corresponding in the plane to, say, the origin. A variant of the argument principle allowing for zeros *on* the curve (see, *e.g.*, [N, p. 131]) implies that γ cannot be simple: the point $0 \in \gamma$ contributes π to the total variation of argument and therefore γ must in addition wind around the origin once.

In either case, let $g : [a, b] \to \mathbb{R}^2$ be the induced parametrization of γ defined on the original interval of definition of f. Since the laying out procedure preserves arc length, then $v_g = |g'| = |f'| = v_f$. Also, the curvature of γ is equal to the curvature of Γ relative to the surface C_{p_0}, and therefore not larger than the curvature of Γ. We conclude from (1) that $S_1 g \leq S_1 f$. Since γ is not simple, it can be subdivided into closed curves γ_1, γ_2 which are differentiable except at the point where γ has self-intersection.
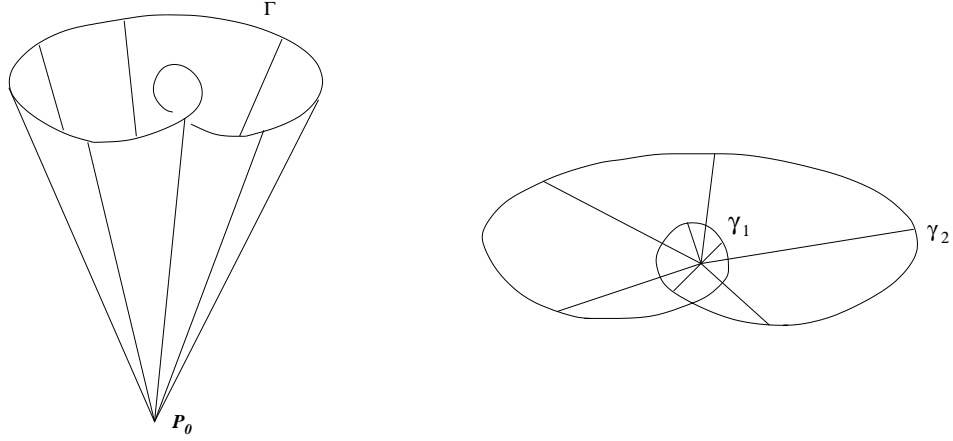
Because g is periodic, one can find intervals $I_1 = [x_1, x_2]$, $I_2 = [x_3, x_4]$ of total length $b - a$ such that

(i) $g_1 = g|_{I_1} : [x_1, x_2] \to \gamma_1$ and $g_2 = g|_{I_2} : [x_3, x_4] \to \gamma_2$;

(ii) the parametrizations g_1, g_2 are C^3 on the open subintervals.

Because g_1, g_2 are closed it follows from Theorem 1 that

$$\int_{x_1}^{x_2} (S_1 g_1)^+ dx > \frac{8}{x_2 - x_1} \quad , \quad \int_{x_3}^{x_4} (S_2 g_2)^+ dx > \frac{8}{x_4 - x_3} \quad .$$

But

$$\frac{1}{x_2 - x_1} + \frac{1}{x_4 - x_3} \geq \frac{4}{(x_2 - x_1) + (x_4 - x_3)} = \frac{4}{b - a},$$

therefore

$$\int_a^b (S_1 f)^+ dx \geq \int_a^b (S_1 g)^+ dx > \frac{32}{b - a},$$

a contradiction.

4. Examples

Example 1: Let $\epsilon > 0$ and fix $I = (-1, 1)$. We seek a non-injective $f : I \to \mathbb{R}^n$ for which

$$\int_I (S_1 f)^+(x) dx < 4 + \epsilon. \tag{11}$$

Let p_n be a sequence of positive, even, C^∞ functions with $\int_I p_n dx = 2 + \epsilon/4$ converging to $p_0 = (2 + \epsilon/4)\delta_0(x)$ in the sense that

$$\lim_{n \to \infty} \int_I p_n h dx = \int_I p_0 h dx$$

for all h continuous on I. For $c = 1/(1 + (\epsilon/8)) < 1$, $u_0(x) = c - |x|$ is a solution of $u'' + p_0 u = 0$ with zeros at $\pm c$. By writing the differential equation $u'' + p_n u = 0$ in integral

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form it is easy to see that, for n sufficiently large, an even solution u_n of $u'' + p_n u = 0$ will have exactly two zeros on I, say at $\pm x_n$. For such an n let

$$g(x) = \int_0^x u_n^{-2}(t)dt \, .$$

This function is odd, increasing and $g(x_n) = -g(-x_n) = \infty$; the intervals $g((x_n - \delta, x_n + \delta))$ and $g((-x_n - \delta, -x_n + \delta))$ overlap. We can avoid the point at infinity by taking $f = T \circ g$ for a suitable Möbius transformation that maps $\mathbb{R} \cup \{\infty\}$ onto a circle in the plane. Again we have an overlap along some arc of the circle. The Schwarzian $S_1 f = S_1 g = Sg$ is left invariant, and a standard calculation shows that $Sg = 2p_n$. Because

$$2 \int_I p_0(x)dx = 4 + \frac{\epsilon}{2}$$

we see that for n sufficiently large,

$$\int_I (S_1 f)^+(x)dx \; < \; 4 + \epsilon \, ,$$

as desired.

Example 2: Fix $J = [-1, 3)$ and $\epsilon > 0$. We seek $f : J \to \mathbb{R}^3$ with a C^3 periodic continuation for which $f(J)$ is a knot and

$$\int_J (S_1 f)^+ dx \; < \; 8 + \epsilon \, . \tag{12}$$

The first step is the construction of a C^3 periodic parametrization $g : J \to \mathbb{R}^2$ satisfying (12) for which $g(J)$ goes around a circle twice. On each half $J_1 = [-1, 1)$ and $J_2 = [1, 3)$ the parametrization g will trace out the circle once in identical fashion. We give the details for J_1. Let $p_0(x)$ be a smooth, non-negative, even approximation of $4\delta(x)$ with support in a small neighborhood of $x = 0$, for which

$$4 \; < \; \int_{J_1} p_0(x)dx \; < \; 4 + \frac{\epsilon}{4} \, . \tag{13}$$

Consider the solution u_0 of

$$u_0'' + p_0 u_0 = 0 \quad , \quad u_0(0) = 1 \, , \; u_0'(0) = 0 \, .$$

Because the optimal constant in (3) for the interval J_1 is 8, we see from (13) that by taking p sufficiently close to the delta function, the function u_0 will have zeros at points $\pm x_0 \in J_1$ close to the endpoints of J_1. Note that u_0 is linear away from the support of p. Consider

$$F(x) = \int_1^x u^{-2}(t)dt \, .$$

Then $F((-x_0, x_0)) = \mathbb{R}$ and $F(-x_0) = F(x_0) = \infty$, so that some Möbius transformation $G = T \circ F$ will map $[-x_0, x_0)$ onto a circle. One can adjust the coefficient p so that $u_0 = 0$ at exactly ± 1. But a linear decay to 0 will prevent the continuation of G to J_2 to be C^3 at $x = 1$. To see this, suppose $u_0(x) = \alpha(1 - x)$ near $x = 1$, and let

$$G = \frac{1}{F} \, .$$

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Then $|G'| = F'/F^2$ is an even function with non-vanishing derivative at $x = 1$, and therefore cannot be continued to J_2 as a smooth periodic function. This shows that we need to modify the asymptotic of u at the zeros in a way that the resulting function $|G'|$ has a vanishing derivative there. For example, near $x = 1$ one can take

$$u(x) = \frac{(1-x) - \frac{1}{3}(1-x^2)^3}{\sqrt{1 - 2x(1-x^2)^2}} \ , \tag{14}$$

which gives near 1

$$F(x) = \frac{1}{(1-x) - \frac{1}{3}(1-x^2)^3} \ .$$

The resulting function $|G'|$ now has vanishing derivative at $x = 1$. A straightforward calculation shows that $u'' + qu = 0$ for a function q that is negative. This means that we can modify the original approximation p_0 to an even function p with

$$\int_{J_1} p^+(x)dx \ < \ 4 + \frac{\epsilon}{4} \ ,$$

for which the solution the even solution of $u'' + pu = 0$ vanishes exactly at ± 1 in accordance with (14). A Möbius transformation of F has the desired can now be extended periodically to J_2 as a smooth parametrization g that goes around a circle C twice. Note that $S_1 g = S_1 F = 2p$, hence

$$\int_J (S_1 g)^+(x)dx \ < \ 8 + \frac{\epsilon}{2} \ .$$

By slightly modifying the curvature and with a small cost in $\int_J (S_1 g)^+(x)dx$, we can arrange that the curve traced out by g restricted to J_2 be almost a circle, disjoint from C except for the point of tangency $g(1)$.

In the final step, we modify g to a parametrization f for which the trace is knotted, once more with negligible cost in $\int_J (S_1 g)^+(x)dx$. This can be accomplished by replacing a small portion of one of the arcs at the point of tangency by a very thin tubular neighborhood, along which the new arc of f will knot around once. This procedure introduces torsion, on which the Schwarzian is independent. It is easy to see that both the modified curvature and velocity remain arbitrarily close to their original values as long as the tubular neighborhood is thin enough. This finishes the example.

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